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Generalization of the h-deformation to higher dimensions

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Abstract. In this article we construct $GL_h(3)$ from $GL_q(3)$ by a singular map. We show that there exist two singular maps which map $GL_q(3)$ to new quantum groups. We also construct their R-matrices and will show that, although the maps are singular, their R-matrices are not. Then we generalize these singular maps to the case $GL(N)$ and for C_n series.

There exist two types of $SL(2)$ quantum groups. One is the standard $SL_q(2)$, the other is the Jordanian quantum group which is also called the h-deformation of $SL(2)$. Quantum matrices in two dimensions, admitting left and right quantum spaces, are classified [1]. One matrix is the q-deformation of $GL(2)$, the other is the h-deformation. The q-deformation of $GL(N)$ has been studied extensively, but in the literature only the two-dimensional case of h-deformation has been studied [2–7].

In [8] it has been shown that $GL_h(2)$ can be obtained from $GL_q(2)$ by a singular limit of a similarity transformation. We will show this method can be used successfully, for construction of $GL_h(N)$. In other words, at first we consider the $GL(3)$ and introduce several singular maps which convert $GL_q(3)$ to $GL_h(3)$. Then we generalize one of the singular maps to the N-dimensional case. We use the R-matrix of $GL_q(N)$ which, by this map, results in a new R-matrix. By this map one can also obtain the h-deformation of the C_n series, but not the B_n and D_n series.

In this article we denote q-deformed objects by primed quantities. Unprimed quantities represent transformed objects.

Consider Manin's q-plane with the following quadratic relation between coordinates:

$$x'y' = qy'x'. \tag{1}$$

Under the following linear transformation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & \frac{h}{q-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2}$$

relation (1) changes to $xy - qyx = hy^2$. For the case of $q = 1$, one obtains the relation for the two-dimensional h-plane. In fact g itself is singular in the $q = 1$ case, but the resulting relation for the plane is non-singular.

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The above linear transformation on the plane induces the following similarity transformation on the R-matrix of $GL_q(2)$:

$$R_h = \lim_{q\to 1}(g \otimes g)^{-1} R_q (g \otimes g). \tag{3}$$

Although the above map is singular, the resulting R-matrix is non-singular and is the well known R-matrix of $GL_h(2)$.

Now consider the three-dimensional Manin's quantum space

$$x'_i x'_j = q x'_j x'_i \qquad i < j \tag{4}$$

and consider the following linear transformation:

$$X = g^{-1} X' \tag{5}$$

where

$$g = \begin{pmatrix} \lambda_1 & \alpha & \beta \\ 0 & \lambda_2 & \gamma \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{6}$$

Here α, β and γ are parameters which can be singular at $q = 1$, so they can be written as $1/f(q)$ where $f(1) = 0$. The Taylor expansion of $f(q)$ about $q = 1$ is $f(q) = (1/h)(q-1) + O((q-1)^2)$. We need only the first term, because we are only interested in the behaviour of $f(q)$ in the neighbourhood of $q = 1$. The coefficient of the first term in the Taylor expansion, h, plays the role of the deformation parameter for the new quantum group. The λ_is can be made equal to 1 by rescaling.

To obtain α, β and γ we should apply this map to the q-deformed plane and its dual, and require that the mapped plane and its dual be non-singular at $q = 1$. The following are the only singular maps satisfying this condition:

$$g_1 = \begin{pmatrix} 1 & \frac{h}{q-1} & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad g_2 = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \frac{h}{q-1} \\ 0 & 0 & 1 \end{pmatrix} \qquad g_3 = \begin{pmatrix} 1 & \alpha & \frac{h}{q-1} \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

Here α, β and γ (in g_1, g_2, g_3), are non-singular parameters. Note that the R-matrices obtained from these maps solve the quantum Yang–Baxter equation and are non-singular for $q = 1$.

Let us denote the dependence of g_1, g_2 and g_3 on parameters explicitly:

$$g_1 := g_1\left(\frac{h}{(q-1)}, \alpha\right) \qquad g_2 := g_2\left(\frac{h}{(q-1)}, \alpha, \beta\right) \qquad g_3 := g_3\left(\frac{h}{(q-1)}, \alpha, \gamma\right). \tag{8}$$

It is easy to show that

$$\begin{aligned} g_1\left(\frac{h}{(q-1)}, \alpha\right) g_1(0, -\alpha) &= g_1\left(\frac{h}{(q-1)}, 0\right) \\ g_2\left(\frac{h}{(q-1)}, \alpha, \beta\right) g_2(0, -\alpha, -\beta) &= g_2\left(\frac{h}{(q-1)}, 0, 0\right) \\ g_3\left(\frac{h}{(q-1)}, \alpha, \gamma\right) g_3(\alpha\gamma, -\alpha, -\gamma) &= g_3\left(\frac{h}{(q-1)}, 0, 0\right) \end{aligned} \tag{9}$$

so all non-singular parameters in the above matrices can be set to zero. Moreover, the R-matrices $R(g_1)$ and $R(g_2)$ which are obtained by formula (3) using $g_1(h/(q-1), 0)$ and $g_2(h/(q-1), 0, 0)$, respectively, are equivalent, because

$$(s \otimes s)^{-1} R(g_2)(s \otimes s) = R(g_1) \tag{10}$$

where $s = e_{13} + e_{21} + e_{32}$. Therefore, there are only two independent cases. The R-matrices corresponding to these transformations are non-singular and have been first obtained by Hietarinta [9]. The first case (the trivial case) is $\alpha = 0$ in g_1 (or $\alpha = \beta = 0$ in g_2) and the second case is $\alpha = \gamma = 0$ in g_3. The h-deformed quantum plane and its dual and R-matrices corresponding to these cases are:

(i) First case.

$$\begin{aligned} &[x_1, x_2] = hx_2^2 \qquad && \eta_3^2 = \eta_2^2 = \{\eta_1, \eta_2\} = 0 \\ &[x_1, x_3] = 0 && \{\eta_2, \eta_3\} = \{\eta_1, \eta_3\} = 0 \\ &[x_2, x_3] = 0 && \eta_1^2 = -h\eta_2\eta_1 \end{aligned} \tag{11}$$

and the non-zero elements of the R-matrix except for $R_{ijij} = 1$ are

$$\begin{aligned} &R_{1121} = R_{2122} = -R_{1112} = -R_{1222} = h \\ &R_{1122} = h^2. \end{aligned} \tag{12}$$

(ii) Second case.

$$\begin{aligned} &[x_1, x_2] = 2hx_3x_2 \qquad && \{\eta_1, \eta_2\} = -2h\eta_3\eta_2 \\ &[x_1, x_3] = hx_3^2 && \eta_1^2 = -h\eta_3\eta_1 \\ &[x_2, x_3] = 0 && \eta_3^2 = \eta_2^2 = \{\eta_1, \eta_3\} = \{\eta_2, \eta_3\} = 0 \end{aligned} \tag{13}$$

and the non-zero elements of R-matrix except for $R_{ijij} = 1$ are

$$\begin{aligned} &R_{1113} = R_{1333} = -h \qquad && R_{1131} = R_{3133} = h \\ &R_{2132} = -R_{1223} = 2h && R_{1133} = h^2. \end{aligned} \tag{14}$$

A linear transformation on the plane induces a similarity transformation on the quantum matrices acting upon it:

$$M' = gMg^{-1}. \tag{15}$$

The algebra of functions, $GL_q(3)$, is obtained from the following relations:

$$R'M_1'M_2' = M_2'M_1'R'. \tag{16}$$

Applying transformation (15) one easily obtains, for the case of $q = 1$,

$$RM_1M_2 = M_2M_1R. \tag{17}$$

So the entries of the transformed quantum matrix M fulfil the commutation relations of the $GL_h(3)$, for both gs. It is easy to show that the h-deformed determinant is central, so it can be set to one. A quantum group's differential structure is completely determined by an R-matrix [10]. One therefore expects that by these similarity transformations the differential structure of the h-deformation be obtained from that of the q-deformation

$$\begin{aligned} &M_2\,\mathrm{d}M_1 - R_{12}\,\mathrm{d}M_1M_2R_{21} = 0 \\ &\mathrm{d}M_2\,\mathrm{d}M_1 + R_{12}\,\mathrm{d}M_1\,\mathrm{d}M_2R_{21} = 0. \end{aligned} \tag{18}$$

Now, it is obvious that defining $\mathrm{d}M := g^{-1}\,\mathrm{d}Mg$ and using the above relations the differential of $GL_h(3)$ can be easily obtained from the corresponding differential structure of $GL_q(3)$.

For the higher dimensions, there are several generalizations which depend on the position of singularity in g. For example we consider the following generalization:

$$g = \sum_{i=1}^{N} e_{ii} + \frac{h}{q-1}e_{1N}. \tag{19}$$

The general aspect of the contraction for arbitrary N can be obtained from this simple map. By inserting this map in (3) we will obtain the general form of the h-deformed R-matrix, which solves the quantum Yang–Baxter equation.

(i) The series A_{n-1}. After applying this singular map, the corresponding h-deformed R-matrix will become

$$R_h = \sum_{i,j=1}^{N} e_{ii} \otimes e_{jj} + 2h \sum_{i>1}^{N-1} (e_{1i} \otimes e_{iN} - e_{iN} \otimes e_{1i}) - h(e_{1N} \otimes e_{NN} - e_{NN} \otimes e_{1N})$$
$$-h(e_{11} \otimes e_{1N} - e_{1N} \otimes e_{11}) + h^2(e_{1N} \otimes e_{1N}). \tag{20}$$

Consider N-dimensional q-deformed quantum space

$$x'_i x'_j = q x'_j x'_i \qquad i < j. \tag{21}$$

Assume the following linear singular transformation

$$x'_i = g_{ij} x_j. \tag{22}$$

By the above transformation and in the $q = 1$ case we obtain the h-deformed quantum plane as follows:

$$\begin{aligned} & x_i x_j = x_j = x_j x_i \qquad 1 < i < j \leqslant N \\ & [x_1, x_j] = 2h x_N x_j \qquad [x_1, x_N] = h(x_N)^2. \end{aligned} \tag{23}$$

(ii) The series B_n, C_n and D_n. The corresponding q-deformed R-matrix has order $N^2 \times N^2$, where $N = 2n+1$ for B_n and $N = 2n$ for D_n and C_n and it is given by [11]:

$$R_q = q \sum_{i \neq i'}^{N} e_{ii} \otimes e_{ii} + e_{\frac{1}{2}(N+1)\frac{1}{2}(N+1)} \otimes e_{\frac{1}{2}(N+1)\frac{1}{2}(N+1)} + \sum_{i \neq j, j'}^{N} e_{ii} \otimes e_{jj} + q^{-1} \sum_{i \neq i'}^{N} e_{i'i'} \otimes e_{ii}$$
$$+(q - q^{-1}) \sum_{i>j}^{N} e_{ij} \otimes e_{ji} - (q - q^{-1}) \sum_{i>j}^{N} q^{\rho_i - \rho_j} \epsilon_i \epsilon_j e_{ij} \otimes e_{i'j'}. \tag{24}$$

The second term is present only for the series B_n. Here $i' = N+1-i$, $j' = N+1-j$, $\epsilon_i = 1$, $i = 1, \ldots, N$ for the series B_n and D_n, $\epsilon_i = 1$, $i = 1, \ldots, (N/2)$, $\epsilon_i = -1$, $i = (N/2)+1, \ldots, N$ for the series C_n and $(\rho_1, \ldots, \rho_N)$ is

$$\begin{aligned} & ((n - \tfrac{1}{2}), \ldots, \tfrac{1}{2}, 0, -\tfrac{1}{2}, \ldots, -n + \tfrac{1}{2}) && \text{for } B_n \\ & (n, n-1, \ldots, 1, -1, \ldots, -n) && \text{for } C_n \\ & (n-1, \ldots, 1, 0, 0, -1, \ldots, -n+1) && \text{for } D_n. \end{aligned} \tag{25}$$

By inserting this R-matrix in (3), the coefficient of $e_{1N} \otimes e_{1N}$ becomes

$$\frac{h^2}{q-1}(q^{-1}+1)(1 + \epsilon_N q^{\rho_N - \rho_1}). \tag{26}$$

This expression is non-singular only when $\epsilon_N = -1$ and for $q = 1$ it is equal to $2Nh^2$. We thus see that only the C_n series remains non-singular. The corresponding h-deformed R-matrix is

$$R_h = \sum_{i,j=1}^{N} e_{ii} \otimes e_{jj} + 2Nh^2 e_{1N} \otimes e_{1N} - 2h \sum_{i=1}^{N} e_{1i} \otimes e_{iN} + \epsilon_i e_{iN} \otimes e_{i'N}$$
$$+2h \sum_{i=1}^{N-1} e_{iN} \otimes e_{1i} - \epsilon_i e_{1i} \otimes e_{1i'}. \tag{27}$$

Therefore by this method we can obtain $SP_h(2n)$. The algebra $SP_q^{2n}(c)$ with generators $x'_1, \ldots, x'_{2n}$ and relations

$$R'_q(x' \otimes x') = qx' \otimes x' \tag{28}$$

is called the algebra of functions on quantum $2n$-dimensional symplectic space. After applying the singular transformation (19) to (28) we obtain the relations between the generators of $SP_h^{2n}(c)$:

$$x_i x_j = x_j x_i \qquad 1 < i < j \leqslant N \qquad j \neq j' \tag{29}$$

$$x_1 x_j = x_j x_1 + 2h x_N x_j \qquad j \neq N$$

$$x_{i'} x_i = x_i x_{i'} + 2h\epsilon_{i'} x_N^2 \qquad 1 < i < i' \leqslant N. \tag{30}$$

In SP_q^{2n} the equality $x''C'x' = 0$ holds. By applying the singular map (29), C' transforms to $C = g^t C' g$, where C is given by

$$C = \sum_{i=1}^{N} \epsilon_i e_{ii'} - Nhe_{NN}. \tag{31}$$

The quantum group $SP_q(2n)$ acts on $SP_q^{2n}(c)$ and preserves $x''C'x' = 0$, so we have

$$M''C'M' = C' \tag{32}$$

and on the other hand

$$M = gM'g^{-1} \qquad M^t = (g^{-1})^t M'' g^t. \tag{33}$$

It follows that

$$M^t CM = C. \tag{34}$$

So we conclude that the quantum group $SP_h(2n)$ acts on $SP_h^{2n}(c)$ and preserves $x^t Cx = 0$. It is interesting to note that the expression $x''C'x'$, which should be equal to one for $SO(2n)$ and $SO(2n+1)$ (B_n and D_n series), is singular. Therefore we cannot obtain the h-deformation of the B_n and D_n series by contraction of the q-deformation, at least by this form (upper triangular matrix) of singular transformation (g).

One of the interesting problems is to construct $U_h(gl(3))$ and its generalization to higher dimensions.

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